PID CONTROLLER SYNTHESIS FOR A CLASS OF UNSTABLE MIMO PLANTS WITH I/O DELAYS [⋆]

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Abstract: Conditions are presented for closed-loop stabilizability of linear time-invariant (LTI) multi-input, multi-output (MIMO) plants with I/O delays (time delays in the input and/or output channels) using PID (Proportional + Integral + Derivative) controllers. We show that systems with at most two unstable poles can be stabilized by PID controllers provided a small gain condition is satisfied. For systems with only one unstable pole, this condition is equivalent to having sufficiently small delay-unstable pole product. Our method of synthesis of such controllers identify some free parameters that can be used to satisfy further design criteria than stability.

Keywords: PID Control, Time Delay, Unstable Systems, MIMO Systems

1. INTRODUCTION

While finite dimensional LTI systems are sufficiently accurate models for a wide range of dynamical phenomena, there are many cases in which delay effects cannot be ignored and have to be included in the model, (Gu et al., 2003). An r input and r output LTI system with I/O delays (time delays in the input and/or output channels) can be represented by $G_\Lambda(s) := \Lambda_o(s)G(s)\Lambda_i(s)$, where G is the finite dimensional part (an $r \times r$ rational matrix), and $\Lambda_\star(s) = \mathrm{diag}\left[e^{-T_1^\star s}, \cdots, e^{-T_r^\star s}\right]$ is the delay matrix, where $\star$ stands for i (input delay case) or o (output delay case). This paper considers closed-loop stabilization (see Fig. 1) of such systems using *proper PID-controllers* (Goodwin et al., 2001):

$$C_{pid}(s) = K_p + \frac{K_i}{s} + \frac{K_d\ s}{\tau_d s + 1}\ , \tag{1}$$

where K_p, K_i, K_d are real matrices and $\tau_d > 0$.

Stability of delay systems of retarded type, or even neutral type, is extensively investigated and many delay-independent and delay-dependent stability results are available, (Gu et al., 2003), (Niculescu, 2001). Also, since delay element is an integral part of process control systems, most of the tuning and internal model control techniques used in process control systems apply to delay systems, (Astrom and Hagglund, 1995). The more special, but practically very relevant problem of existence of stabilizing PID-controllers is unfortunately not easy to solve even for the delay-free case. One way of gaining insight into the difficulty of the problem is to note that the ex-

[⋆] This work was supported in part by the European Commission (contract no. MIRG-CT-2004-006666) and by TÜBİTAK (grant no. EEEAG-105E065 and BAYG).

istence of a stabilizing PID-controller for a plant of transfer matrix $G(s)$ is equivalent to that of a *constant stabilizing output feedback* for a transformed MIMO plant (at this point an LMI approach can be used, see e.g. (Lin et al., 2004) and the references therein). Alternatively, the problem can be posed as determining conditions of existence of a stable and fixed-order controller for the extended plant $G(s)\frac{s+1}{s}$, which is again well-known to be a difficult problem, (Blondel et al., 1994; Vidyasagar, 1985). It should be mentioned that there *are* some computational PID-stabilization methods, which consist of "efficient search" in the parameter space, recently developed for single-input single-output (SISO) delay-free systems (see (Saadaoui and Ozguler, 2005) and the references therein). Some of these techniques have been extended to cover scalar, single-delay systems,(Silva et al., 2005).

In this paper, making a novel use of the small gain theorem, we obtain two main results: First, for MIMO plants with input and/or output delays, we obtain some sufficient conditions on the existence of stabilizing PID controllers, and second, we explicitly construct PID controllers for plants having only one unstable pole (under the condition that the product of the unstable pole with delay is sufficiently small). This construction is extended to the case of two unstable real or complex poles. As our goal is to establish existence of stabilizing PID controllers at this point, we do not consider performance issues but propose freedom in the design parameters that can be used to satisfy performance criteria.

Notation: As usual, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}_-$, $\mathbb{C}_+$ denote real, complex, open left-half plane complex and open right-half plane complex numbers; $\mathcal{U}$ denotes the extended closed right-half plane, i.e., $\mathcal{U} = \{\, s \in \mathbb{C} \mid \mathcal{R}e(s) \geq 0 \,\} \cup \{\infty\}$; $\mathbf{R_p}$ denotes proper rational functions; $\mathbf{S}$ denotes stable proper real rational functions of s. The set of matrices whose entries are in $\mathbf{S}$ is denoted by $\mathcal{M}(\mathbf{S})$. The space $\mathcal{H}_\infty$ is the set of all bounded analytic functions in $\mathbb{C}_+$. For $h \in \mathcal{H}_\infty$, the norm is defined as $\|h\|_\infty = \text{ess}\sup_{s\in\mathbb{C}_+}|h(s)|$, where ess sup denotes the essential supremum. A matrix-valued function H is in $\mathcal{M}(\mathcal{H}_\infty)$ if all its entries are in $\mathcal{H}_\infty$, and in this case $\|H\|_\infty = \text{ess}\sup_{s\in\mathbb{C}_+}\overline{\sigma}(H(s))$, where $\bar{\sigma}$ denotes the maximum singular value. From the induced L^2 gain point of view, a system with transfer matrix H is stable if and only if $H \in \mathcal{M}(\mathcal{H}_\infty)$. Moreover, for square $H \in \mathcal{M}(\mathcal{H}_\infty)$, we say that H is unimodular if $H^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$. For simplicity, we drop (s) in transfer matrices such as $G(s)$ where this causes no confusion. Also, since all norms we are interested in are $\mathcal{H}_\infty$ norms, we will drop the norm subscript, i.e. $\|\cdot\|_\infty \equiv \|\cdot\|$ whenever this is clear from the context.

2. PROBLEM DESCRIPTION

Consider the standard unity-feedback system shown in Fig. 1, where $G \in \mathbf{R_p}^{r\times r}$ and $C \in \mathbf{R_p}^{r\times r}$ denote the plant without the time delay term (non-delayed plant, for short) and the controller transfer-functions. It is assumed that the feedback system is well-posed and that the non-delayed plant and the controller have no unstable hidden-modes. It is also assumed that $G \in \mathbf{R_p}^{r\times r}$ is full normal rank. The delay terms are in the form $\Lambda_\star = \text{diag}\left[e^{-sT_1^\star}, \ldots, e^{-sT_r^\star}\right]$, where, for $1 \leq j \leq r$, we have $T_j^\star \in \Theta_j^\star = [0\ ,\ T_{j,\max}^\star) \subset \mathbb{R}_+$ and $\star$ stands for i (input channel delays) or o (output channel delays). We assume that the delay upper bound $T_{j,\max}^\star$ is known for all input and output channels $j = 1, \ldots, r$. Define $\mathcal{T}^\star := (T_1^\star, \ldots, T_r^\star)$ and $\Theta^\star := (\Theta_1^\star, \ldots, \Theta_r^\star)$. As a shorthand notation we will write $(\mathcal{T}^i, \mathcal{T}^o) =: \mathcal{T} \in \Theta := (\Theta^i, \Theta^o)$ to represent all possibilities $T_j^\star \in \Theta_j^\star$, $1 \leq j \leq r$. We denote the delayed plant by $G_\Lambda := \Lambda_o(s)G(s)\Lambda_i(s)$. The closed-loop transfer matrix H_{cl} from (r, v) to (u, y) is

$$H_{cl} = \begin{bmatrix} C(I + G_\Lambda C)^{-1} & -C(I + G_\Lambda C)^{-1}G_\Lambda \\ G_\Lambda C(I + G_\Lambda C)^{-1} & (I + G_\Lambda C)^{-1}G_\Lambda \end{bmatrix}. \quad (2)$$

We consider the proper form of PID-controllers in (1), where the real matrices K_p, K_i, K_d are called the proportional constant, the integral constant, and the derivative constant, respectively. Due to implementation issues of the derivative action, a pole is typically added to the derivative term (with $\tau_d \in \mathbb{R}$, $\tau_d > 0$ when $K_d \neq 0$) so that the transfer-function C_{pid} in (1) is proper. If one or more of the three terms K_p, K_i, K_d is zero, then the corresponding subscript is omitted from C_{pid}.

Definition 1. a) The feedback system $Sys(G_\Lambda, C)$, shown in Fig. 1, is said to be stable iff the closed-loop map H_{cl} is in $\mathcal{M}(\mathcal{H}_\infty)$. b) A delayed plant G_Λ, where $G \in \mathbf{R_p}^{r\times r}$, is said to admit a PID-controller iff there exists a PID-controller $C = C_{pid}$ as in (1) such that the system $Sys(G_\Lambda, C)$ is stable. We say that G_Λ is stabilizable by a PID-controller, and C_{pid} is a stabilizing PID-controller. □

Let $G = Y^{-1}X$ be any left coprime factorization (LCF) of the plant, $C = N_c D_c^{-1}$ be any right coprime factorization (RCF) of the controller, where we use coprime factorizations over $\mathbf{S}$; i.e., for $G \in \mathbf{R_p}^{r\times r}$, $X, Y \in \mathcal{M}(\mathbf{S})$ and $\det Y(\infty) \neq 0$, and similarly for $C \in \mathbf{R_p}^{r\times r}$, $N_c, D_c \in \mathcal{M}(\mathbf{S})$ and $\det D_c(\infty) \neq 0$. Let X_Λ denote the "numerator" matrix of G_Λ, i.e., $X_\Lambda := \Lambda_o(s)X(s)\Lambda_i(s)$. Now if the "denumerator" matrix Y of $G = Y^{-1}X$ is diagonal, then the delayed plant G_Λ can be

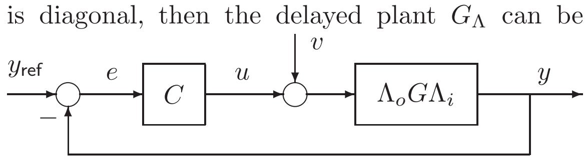

Fig. 1. Unity-feedback system $Sys(G_\Lambda, C)$.

expressed as $G_\Lambda = Y^{-1}X_\Lambda$. The controller C stabilizes G_Λ if and only if $M_\Lambda := YD_c + X_\Lambda N_c \in \mathcal{M}(\mathcal{H}_\infty)$ is unimodular, i.e., $M_\Lambda^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$, (Smith, 1989).

3. MAIN RESULTS

Throughout the paper we assume that Y^{-1} is diagonal, hence it commutes with Λ_o. Thus $G_\Lambda = Y^{-1}X_\Lambda$ in all cases studied here.

The result in Lemma 1 below will be used in designing PI or PID controllers from P or PD controllers, i.e., integral action will be added once P and D terms are designed. This result is a slight extension of Theorem 5.3.10 of (Vidyasagar, 1985) to systems with time delays.

Lemma 1. (Two-step controller synthesis): Let $G \in \mathbf{R_p}^{r\times r}$. Suppose that C_g is a controller that stabilizes G_Λ, and C_h is a controller that stabilizes the stable system $H_\Lambda := G_\Lambda(I + C_g G_\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$. Then $C = C_g + C_h$ is also a controller that stabilizes G_Λ. □

Although it is obvious that stable plants admit PID-controllers, the freedom in the stabilizing controller parameters is still worth investigating. We propose a PID-controller synthesis for stable plants in Proposition 2 below, which will be frequently referred to in the sequel.

Proposition 2. (PID-controller synthesis for stable plants): Let $G \in \mathbf{S}^{r\times r}$ and assume (normal) rank$G(s) = r$. *i) PD-design*: Choose any $\hat{K}_p\ \hat{K}_d \in \mathbb{R}^{r\times r}$, $\tau_d > 0$. Define $\widehat{C}_{pd} := \hat{K}_p + \frac{\hat{K}_d s}{\tau_d s+1}$. Then, for any α satisfying $0 < \alpha < \|G\,\widehat{C}_{pd}\|^{-1}$ a PD-controller that stabilizes G_Λ for $\mathcal{T} \in \Theta$ is

$$C_{pd}(s) = \alpha\,\widehat{C}_{pd} = \alpha\,\hat{K}_p + \frac{\alpha\,\hat{K}_d s}{\tau_d s + 1}. \qquad (3)$$

ii) PID-design: Let rank$G(0) = r$. Choose any $\hat{K}_p,\ \hat{K}_d \in \mathbb{R}^{r\times r}$, $\tau_d > 0$. Define $\widehat{C}_{pid} = \hat{K}_p + \frac{G(0)^{-1}}{s} + \frac{\hat{K}_d s}{\tau_d s+1}$. Then, for any γ satisfying

$$0 < \gamma < \max\{\min_{\mathcal{T}\in\Theta} \|\Psi\|^{-1}, \min_{\mathcal{T}\in\Theta} \|\widetilde{\Psi}\|^{-1}\}, \qquad (4)$$

where $\Psi = \frac{sG_\Lambda(s)\hat{C}_{pid} - I}{s}$, $\widetilde{\Psi} = \frac{s\hat{C}_{pid}G_\Lambda(s) - I}{s}$, a PID-controller stabilizing G_Λ for $\mathcal{T} \in \Theta$ is

$$C_{pid}(s) = \gamma\,\widehat{C}_{pid}. \quad \square \qquad (5)$$

Proposition 3 below gives general existence conditions for stabilizing PID controllers. If a stabilizing P, I, or D-controller exists, then it can be extended to a stabilizing PI, ID, PD, PID-controller:

Proposition 3. (General existence conditions for stabilizing PID-controllers): Let $G \in \mathbf{R_p}^{r\times r}$. Let (normal) rank$G(s) = r$. **a)** If G_Λ admits a PID-controller such that the integral constant $K_i \in \mathbb{R}^{r\times r}$ is nonzero, then G has no transmission-zeros at $s = 0$ and rank$K_i = r$. **b)** If G_Λ admits a PID-controller such that any one of the three constants K_p, K_d, K_i is nonzero, then G_Λ admits a PID-controller such that any two of the three constants is nonzero, and G_Λ admits a PID-controller such that all of the three constants is nonzero. **c)** If G_Λ admits a PID-controller such that two of the three constants K_p, K_d, K_i is nonzero, then G_Λ admits a PID-controller such that all of the three constants is nonzero. In **b)** and **c)**, the integral constant $K_i \neq 0$ only if G has no transmission-zeros at $s = 0$. □

Proposition 3 does not explicitly define which plant classes admit P, I, or D-controllers. We investigate specific classes of plants and propose stabilizing PID-controller design methods next in Section 3.1.

3.1 Delayed plants that admit PID-controllers

Lemma 2. (Strong stabilizability is a necessary condition for PID stabilization): Let $G \in \mathbf{R_p}^{r\times r}$. Let rank$G(s) = r$. If G_Λ admits a PID-controller for any $\mathcal{T} \in \Theta$, then G is strongly stabilizable. □

We now consider plants with a limited number of $\mathcal{U}$-poles, including the origin. Such limitations on the number of $\mathcal{U}$-poles are not surprising. Clearly, plants with an odd number of positive real-axis poles are not even strongly stabilizable if there are two or more positive real-axis zeros (including infinity). But even when the parity-interlacing-property is satisfied, plants that have more than two $\mathcal{U}$-poles do not necessarily admit PID-controllers. For example, by using the Routh-Hurwitz test it can easily be shown that the plant $(s-p)^{-3}$ does not admit a stabilizing PID controller for $p \geq 0$.

3.1.1. Plants with only one unstable real-axis pole

We consider transfer matrices G in the form

$$G = Y^{-1}X = [\ \frac{(s-p)}{as+1} I\ ]^{-1}\ [\ \frac{(s-p)}{as+1} G\ ]\ , \quad (6)$$

where $p \in \mathbb{R}$, $p \geq 0$ and $a \in \mathbb{R}$, $a > 0$, and rank$X(p) = $ rank$(s-p)G(s)|_{s=p} = r$. Furthermore, since G has no transmission-zeros at $s = 0$, rank$X(0) = $ rank$(s-p)G(s)|_{s=0} = r$. In this paper, by a slight abuse of notation, we say that G has only one unstable pole if $Y(s)$, in (6), is identity times a scalar transfer function with a single zero in the closed right half plane.

Proposition 4. Let $G \in \mathbf{R_p}^{r\times r}$, be as in (6), with $X = \frac{(s-p)}{as+1}G \in \mathcal{M}(\mathbf{S})$, rank$X(p) = r$. Let $X(0)$ be nonsingular, $G^{-1}(0) = -p\,X(0)^{-1}$. *i) PD-design*: Choose any $\hat{K}_d \in \mathbb{R}^{r\times r}$, $\tau_d > 0$. Define $\widehat{C}_{pd} := X(0)^{-1} + \frac{\hat{K}_d s}{\tau_d s+1}$ and $\Phi_\Lambda := \frac{(s-p)G_\Lambda(s)\,\widehat{C}_{pd}(s) - I}{s}$, $\widetilde{\Phi}_\Lambda := \frac{\widehat{C}_{pd}(s)(s-p)G_\Lambda(s) - I}{s}$. If $0 \leq p < \max\{\min_{\mathcal{T}\in\Theta} \|\Phi_\Lambda\|^{-1}, \min_{\mathcal{T}\in\Theta} \|\widetilde{\Phi}_\Lambda\|^{-1}\}$, then for any positive $\alpha \in \mathbb{R}$ satisfying (7), a PD-controller that stabilizes G_Λ for $\mathcal{T} \in \Theta$ is given by (8); if $\hat{K}_d = 0$, (8) is a P-controller:

$$p < \alpha + p < \max\{\min_{\mathcal{T}\in\Theta} \|\Phi_\Lambda\|^{-1}, \min_{\mathcal{T}\in\Theta} \|\widetilde{\Phi}_\Lambda\|^{-1}\}, \quad (7)$$

$$C_{pd}(s) = (\,\alpha + p\,)\,\widehat{C}_{pd}(s). \qquad (8)$$

ii) PID-design: Let C_{pd} be as in (8). Let $H_{pd} := G_\Lambda(I + C_{pd}G_\Lambda)^{-1}$. Then for any $\gamma \in \mathbb{R}$ satisfying

(9), a PID-controller that stabilizes G_Λ for $\mathcal{T} \in \Theta$ is given by (10) where $H_{pd}(0)^{-1} = \alpha\, X(0)^{-1}$; if $\hat{K}_d = 0$, is a PI-controller:

$$0 < \gamma < \max\{\min_{\mathcal{T}\in\Theta} \|\Upsilon\|^{-1}, \min_{\mathcal{T}\in\Theta} \|\widetilde{\Upsilon}\|^{-1}\}, \quad (9)$$

where $\Upsilon = \frac{H_{pd}(s)H_{pd}(0)^{-1}-I}{s}$, $\widetilde{\Upsilon} = \frac{H_{pd}(0)^{-1}H_{pd}(s)-I}{s}$

$$C_{pid}(s) = C_{pd}(s) + \frac{\gamma\,\alpha\, X(0)^{-1}}{s}\,. \qquad \square \quad (10)$$

Example 1. Consider the delayed plant $G_\Lambda(s) = \frac{e^{-sT}}{s-p}$, where $p > 0$. Then for $a > 0$, $X := 1/(as+1)$, $X(0) = 1$. Choose any $\hat{K}_d \in \mathbb{R}$, $\tau_d > 0$. By Proposition 4, if $p < \min_{\mathcal{T}\in\Theta} \|\Phi_\Lambda\|^{-1} = \min_{\mathcal{T}\in\Theta} \|\frac{e^{-sT}-1}{s} + e^{-sT}\frac{\hat{K}_d}{\tau_d s+1}\|^{-1}$, then for any α as in (7), $C_{pd}(s) = (p+\alpha) + \frac{(p+\alpha)\ \hat{K}_d s}{\tau_d s+1}$ is a stabilizing PD-controller for G_Λ. Note that for SISO plants, $\Phi_\Lambda = \tilde{\Phi}_\Lambda$. Now consider proportional controller design for a fixed T and p in this example. It is easy to show that a stabilizing P-controller exists if and only if $pT < 1$. Moreover, for any fixed $pT < 1$, there is a maximum allowable gain $K_{\max}$ for the proportional controller; this is shown in Fig. 2 (a) as the exact bound. On the other hand, our approach uses the small gain argument and leads to $C_p = (p+\alpha)$ as the controller gain. With $\|\Phi_\Lambda\| = \|T\ \frac{(e^{-sT}-1)}{sT}\| = T$, the condition $p < \|\Phi_\Lambda\|^{-1}$ is the same as $pT < 1$. From the bound given in (7), $\alpha < T^{-1} - p$; the largest controller gain we can use in our case is $1/T$. This bound is also shown in Fig. 2 (a), which illustrates that the approach used here is not too conservative. Fig. 2 (a) also demonstrates the difficulty of controlling this plant using a proportional controller when the product of the unstable pole with delay is relatively large. Other fundamental performance limitations can also be quantified in terms of smallest achievable sensitivity level, (Stein, 1989), or mixed sensitivity $\mathcal{H}_\infty$ cost, (Enns et al., 1992). It is also clear that by using the derivative term we can improve the bound on largest allowable pT. The largest pole delay product for which we can find a PD-controller is $1.38 = 1/0.725$, and that corresponds to $\tau_d \to 0$ and $\hat{K}_d/T = 0.31$. $\square$

Example 2. Consider the transfer matrix $G(s)$ of a distillation column, (Friedland, 1986), where $G(s) = \frac{1}{s}\, G_o\, G_1(s)$ with $G_o = \begin{bmatrix} 3.04 & -\frac{278.2}{180} \\ 0.052 & \frac{206.6}{180} \end{bmatrix}$ and $G_1(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{180}{(s+6)(s+30)} \end{bmatrix}$. An LCF of the plant is $G(s) = Y(s)^{-1}X(s)$, with $X(s) = \frac{1}{as+1}G_oG_1(s)$, $Y(s) = \frac{s}{as+1}I$, $a > 0$. Assume that the delays in the input channels are h_1 and h_2, and consider proportional control only. In this case we have $\widehat{C}_p = X(0)^{-1} = G_o^{-1}$, $C_p(s) = \alpha X(0)^{-1} = \alpha G_o^{-1}$, and $\Phi_\Lambda(s) = (\frac{G_o\ G_1(s)\Lambda_i(s)\ G_o^{-1} - I}{s})$. Fig. 3 shows $\|\Phi_\Lambda\|^{-1}$ versus h_1 and h_2, from which we see that the largest value 4.86 is obtained for

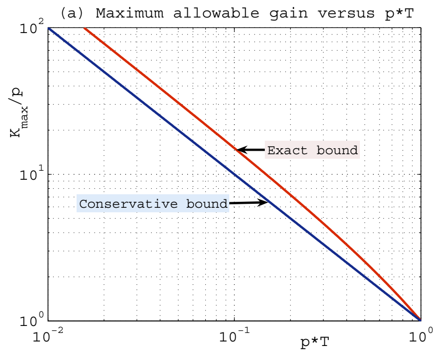

Fig. 2. Maximum K_p versus pT.

$h_1 = 0.18$ and $h_2 = 0$. Note that 0.18 sec delay is needed in the first channel to equalize the phase lag in the input channels of $G_1\Lambda_i$. In this case

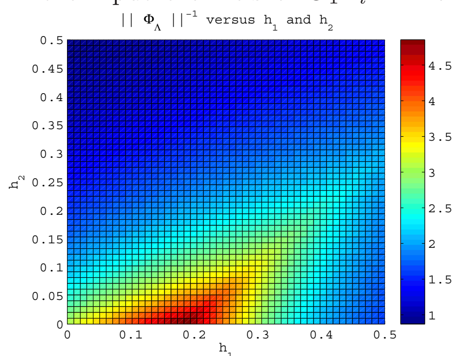

Fig. 3. Maximum $\|\Phi_\Lambda\|^{-1}$ versus h_1 and h_2.

stability is guaranteed if $\alpha < \|\tilde{\Phi}_\Lambda\|^{-1}$, where $\|\tilde{\Phi}_\Lambda\| = \max\{h_1\ ,\ h_2 + 0.2\}$. Clearly, the largest gain allowable is $\alpha_{\max} = 5$, for $h_2 = 0$ and $0 \le h_1 < 0.2$. This result is less conservative than the one obtained using the bound $\alpha < \|\Phi_\Lambda\|^{-1}$. Note that for $h_2 = 0$, and $h_1 > 0.2$ we have $\alpha_{\max} = 1/h_1$. But, when $C(s) = \alpha G_o^{-1}$, the characteristic equation of this system is $(1 + \frac{\alpha\ e^{-h_1 s}}{s})(1 + \frac{\alpha\ 180e^{-h_2 s}}{s(s+6)(s+30)}) = 0$. When $h_2 = 0$, actual largest gain we can use is $\alpha_{\max,act} = \min\{\alpha_{\max,1}\ ,\ 36\}$, where $\alpha_{\max,1} = \frac{\pi}{2h_1}$, and for $h_1 > 0.2$ we have

$$\alpha_{\max,act} = \frac{\pi}{2h_1} \approx \frac{1.57}{h_1} > \alpha_{\max} = \frac{1}{h_1}. \quad (11)$$

The level of conservatism in this example is characterized by (11). Now consider the PD-controller $C_{pd} = \alpha(I + \frac{\tilde{K}_d\, s}{\tau_d s+1})G_o^{-1}$ in (8), where $\hat{K}_d =: \tilde{K}_d G_o^{-1}$. The optimal derivative gain matrix $\hat{K}_d = \tilde{K}_d G_o^{-1}$ is the one which minimizes $\|\tilde{\Phi}_\Lambda\|$. Since $\tilde{\Phi}_\Lambda$ is diagonal, we restrict $\tilde{K}_d$ to be in the form $diag(K_{d,1}, K_{d,2})$. Fig. 4 shows optimal $K_{d,1}$ (resp. $K_{d,2}$) versus h_1 (resp. h_2). $\square$

3.1.2. Plants with two unstable poles Let $G(s) \in \mathbf{R_p}^{r\times r}$ have full (normal) rank. Let G have no transmission-zeros at $s = 0$. Define $d := (a_1 s + 1)(a_2 s + 1)$ and $n := (s - p_1)(s - p_2)$, where $p_1, p_2 \in \mathcal{U}$, $a_1, a_2 \in \mathbb{R}$, $a_1, a_2 > 0$, and let G have an LCF $G = Y^{-1}X$ of the form

$$G = Y^{-1}X = [\ \frac{n}{d}I\ ]^{-1}\ [\ \frac{n}{d}G\ ]\,, \quad (12)$$

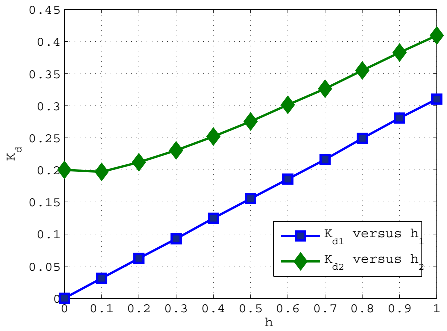

Fig. 4. Optimal $K_{d,1}$ and $K_{d,2}$.

where $\text{rank}X(p_j) = \text{rank}\, nG(s)|_{s=p_j} = r\,,\, j = 1, 2$. Furthermore, since G has no transmission-zeros at $s = 0$, $\text{rank}X(0) = \text{rank}\, nG(s)|_{s=0} = r$. We consider real and complex-conjugate pairs of poles as two separate cases:

Case a) The two unstable poles are real, i.e., $p_j \in \mathbb{R}$, $p_j \geq 0$, $j = 1, 2$. Proposition 5-(a) shows that under certain assumptions, the delayed plant G_Λ admits PD and PID-controllers. Some plants in this class (for example, $G = \frac{1}{(s-p_1)(s-p_2)}$, $p_1 \geq 0, p_2 \geq 0$) do not admit P, D, or I-controllers.

Case b) The two poles are a complex-conjugate pair, i.e., $p_1 = \bar{p}_2\,,\, n = s^2-(p_1+p_2)s+p_1p_2 = s^2-2fs + g^2$, $f \geq 0$, $g > 0$, $f < g$. In this case, $X(0) = g^2G(0)$. Proposition 5-(b) shows that under certain assumptions, the delayed plant G_Λ admits D, PD, ID, PID-controllers. Some plants in this class (for example, $G = \frac{1}{s^2+g^2}$, $g \geq 0$) do not admit P-controllers or I-controllers.

Proposition 5. Let G be as in (12), with $X = \frac{n}{d}G \in \mathbf{S}^{r\times r}$, $\text{rank}X(p_j) = r$, $j = 1, 2$. Let $X(0)$ be nonsingular. Choose any $\tau_d > 0$. Define $\Phi_\Lambda := s^{-1}\left(\frac{n}{(\tau_d s+1)}G_\Lambda(s)\, X(0)^{-1} - I\right)$, $\tilde{\Phi}_\Lambda := s^{-1}\left(\frac{n}{(\tau_d s+1)}X(0)^{-1}G_\Lambda(s) - I\right)$. **a)** Let $p_j \in \mathbb{R}$, $p_j \geq 0, j = 1, 2$. *i) PD-design*: If $0 \leq p_1 < \Omega$ where $\Omega := \max\{\min_{\mathcal{T}\in\Theta} \|\Phi_\Lambda\|^{-1}, \min_{\mathcal{T}\in\Theta} \|\widetilde{\Phi}_\Lambda\|^{-1}\}$, then choose any $\alpha_1 \in \mathbb{R}$ satisfying

$$p_1 < \alpha_1 + p_1 < \Omega. \tag{13}$$

Define $W := (s - p_2)G_\Lambda(s)X(0)^{-1}$ and $\widetilde{W} = (s - p_2)X(0)^{-1}G_\Lambda(s)$. Let $\Phi_{2\Lambda} := \frac{\alpha_1(I+\frac{(\alpha_1+p_1)}{\tau_d s+1}W)^{-1}W-I}{s}$, $\widetilde{\Phi}_{2\Lambda} := \frac{\alpha_1(I+\frac{(\alpha_1+p_1)}{\tau_d s+1}\widetilde{W})^{-1}\widetilde{W}-I}{s}$. If $0 \leq p_2 < \Omega_2$, where $\Omega_2 := \max\{\min_{\mathcal{T}\in\Theta} \|\Phi_{2\Lambda}\|^{-1}, \min_{\mathcal{T}\in\Theta} \|\widetilde{\Phi}_{2\Lambda}\|^{-1}\}$, then choose any $\alpha_2 \in \mathbb{R}$ satisfying

$$p_2 < \alpha_2 + p_2 < \Omega_2. \tag{14}$$

Let $K_p = (\alpha_1\alpha_2 - p_1\, p_2)X(0)^{-1}$, $K_d = (\alpha_1 + p_1\,)\,(1 + \tau_d\, p_2)X(0)^{-1}$; then a PD-controller that stabilizes G_Λ for $\mathcal{T} \in \Theta$ is given by $C_{pd}(s) = K_p+ \frac{K_d s}{\tau_d s+1}$. *ii) PID design*: Let C_{pd} be as above. Then for any $\gamma \in \mathbb{R}$ satisfying (9) with $H_{pd}(0)^{-1} = \alpha_1\, \alpha_2\, X(0)^{-1}$, a PID-controller that stabilizes G_Λ for $\mathcal{T} \in \Theta$ is given by (15):

$$C_{pid}(s) = C_{pd}(s) + \frac{\gamma\, \alpha_1\, \alpha_2\, X(0)^{-1}}{s}\ . \tag{15}$$

b) Let $p_1 = \bar{p}_2 \in \mathbb{C}$, $n = s^2 - (p_1 + p_2)s + p_1p_2 = s^2 - 2fs + g^2$, $f \geq 0$, $g > 0$, $f < g$. *i) PD-design*: If $f + 2g < \Omega$, then choose any $\beta_1, \beta_2 \in \mathbb{R}$, $\beta_1, \beta_2 \geq 0$, satisfying

$$\beta_1 + \beta_2 + (f + 2g) < \Omega. \tag{16}$$

Let $K_p = [\beta_1\beta_2 + \beta_1(g - f) + \beta_2 g - fg\,]X(0)^{-1}$, $K_d = (\beta_1 + \beta_2 + f + 2g)X(0)^{-1} - \tau_d\, K_p$; then a PD-controller that stabilizes G_Λ for $\mathcal{T} \in \Theta$ is

$$C_{pd}(s) = K_p + \frac{K_d\, s}{\tau_d s + 1} = \vartheta(s)\frac{G(0)^{-1}}{g^2} \tag{17}$$

$$\vartheta := \frac{(\beta_1 + \beta_2 + f + 2g)s + \beta_1(\beta_2 + g - f) + \beta_2 g - fg}{\tau_d\, s + 1}.$$

If $2(f + g) < \Omega$, let $K_d = 2(f + g)X(0)^{-1}$; then a D-controller that stabilizes G_Λ is

$$C_d(s) = \frac{K_d\, s}{\tau_d s + 1} = \frac{2\,(f + g)\,)\, G(0)^{-1}\, s}{g^2\,(\tau_d\, s + 1)}\ . \tag{18}$$

ii) PID-design: Let C_{pd} be as in (17). Then for any $\gamma \in \mathbb{R}$ satisfying (9) with $H_{pd}(0)^{-1} = (\beta_1 + g)(\beta_2 + g - f)\, X(0)^{-1}$, a PID-controller that stabilizes G_Λ for $\mathcal{T} \in \Theta$ is

$$C_{pid}(s) = C_{pd}(s) + \frac{\gamma(\beta_1 + g)(\beta_2 + g - f)}{s}\frac{G(0)^{-1}}{g^2}. \tag{19}$$

Let C_d be as in (18). Then for any $\gamma \in \mathbb{R}$ satisfying (9) with $H_d(0)^{-1} = g^2\, X(0)^{-1} = G^{-1}(0)$, an ID-controller that stabilizes G_Λ for $\mathcal{T} \in \Theta$ is

$$C_{id}(s) = C_d(s)\ + \frac{\gamma\, G(0)^{-1}}{s}\ . \qquad \square \tag{20}$$

4. CONCLUSIONS

We showed existence of stabilizing PID-controllers for a class of LTI, MIMO plants with delays in the input and/or output channels. Moreover, for plants with only one or two unstable poles (and finitely many $\mathbb{C}_-$ poles) we gave explicit formulae for PID controller parameters. These results are obtained from a small gain based argument. Therefore, they are conservative. We were able to quantify the level of conservatism on the examples given.

In the light of inequality conditions (7) and (9) of Proposition 4, an interesting problem to study is the computation of optimal $\widehat{K}_d$ which minimizes $\|\Phi\|$ or $\|\widetilde{\Phi}\|$, and optimal $\alpha, \widehat{K}_d$ minimizing $\|\Upsilon\|$ or $\|\widetilde{\Upsilon}\|$. Figure 4 answers this question partially for the specific example considered. The numerical values in this figure are computed from a brute-force search. An analytic solution is possible, see (Ozbay and Gundes, 2006) for further details.

APPENDIX: PROOFS

Proof of Lemma 1: Let $G = Y^{-1}X$ be an LCF; let $C_g = N_gD_g^{-1}$ be an RCF. The controller $C_g = N_gD_g^{-1}$ stabilizes $G_\Lambda = Y^{-1}X_\Lambda$ if and only if $M_\Lambda := YD_g + X_\Lambda N_g$ is unimodular. Since C_g stabilizes G, the transfer-functions $H_\Lambda = G_\Lambda(I + C_gG_\Lambda)^{-1}$ and $I - C_gH_\Lambda = (I + C_gG_\Lambda)^{-1}$ are stable. Now C_h stabilizes $H_\Lambda \in \mathcal{M}(\mathcal{H}_\infty)$ if and only if $C_h(I + H_\Lambda C_h)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$, and $(I + H_\Lambda C_h)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$. Write $C = C_g + C_h = [N_g + (I - C_gH_\Lambda)C_h(I + H_\Lambda C_h)^{-1}D_g\,]\,[(I + H_\Lambda C_h)^{-1}D_g]^{-1}$. Define $N_c := [N_g + (I - C_gH_\Lambda)C_h(I + H_\Lambda C_h)^{-1}D_g\,] \in \mathcal{M}(\mathcal{H}_\infty)$, $D_c := (I + H_\Lambda C_h)^{-1}D_g \in \mathcal{M}(\mathcal{H}_\infty)$. Then $YD_c + X\Lambda N_c = Y[(I+H_\Lambda C_h)^{-1}+H_\Lambda C_h(I+H_\Lambda C_h)^{-1}]D_g+X_\Lambda N_g = M_\Lambda$ is unimodular. Therefore, $C = N_cD_c^{-1}$ is a stabilizing controller for G_Λ. $\square$

Proof of Proposition 2: *i)* Let $M_{pd} := I + G_\Lambda\, C_{pd} = I + \alpha G_\Lambda\, \widehat{C}_{pd}$; then M_{pd} is unimodular since $\alpha \parallel G_\Lambda\, \widehat{C}_{pd} \parallel = \alpha \parallel G\ \ \widehat{C}_{pd} \parallel < 1$. Therefore, C_{pd} stabilizes G_Λ. Since $\hat{K}_p$, $\hat{K}_d$ are arbitrary, they can be zero. *ii)* The controller C_{pid} stabilizes G_Λ if and only if $M_{pid} := \frac{s}{s+\gamma} I + G_\Lambda \frac{s}{s+\gamma}\, C_{pid}$ is unimodular, and equivalently $\widetilde{M}_{pid} := \frac{s}{s+\gamma} I + \frac{s}{s+\gamma}\, C_{pid}\, G_\Lambda$ is unimodular. Writing $M_{pid} = I + \frac{\gamma\, s}{s+\gamma}\, \frac{(s\, G_\Lambda(s)\ \widehat{C}_{pid} - I)}{s}$, a sufficient condition for M_{pd} to be unimodular is that γ satisfies the first upper bound in (4). Similarly, writing $\widetilde{M}_{pid} = I + \frac{s}{s+\gamma}\, \frac{(s\ \widehat{C}_{pid}\, G_\Lambda(s) - I)}{s}$, a sufficient condition for $\tilde{M}_{pid}$ to be unimodular is that γ satisfies the second upper bound in (4). Since M_{pid} is unimodular if and only if $\tilde{M}_{pid}$ is unimodular, the less conservative one of these bounds suffices and hence, C_{pid} in (5) stabilizes G_Λ for $\gamma \in \mathbb{R}$ satisfying (4). □

Proof of Proposition 3: **a)** Let $G = Y^{-1}X$ be an LCF of G. Let $C_{pid} = K_p + \frac{K_i}{s} + \frac{K_d\, s}{\tau_d s+1}$ be a PID-controller that stabilizes G_Λ. For any positive $a \in \mathbb{R}$, an RCF $C_{pid} = N_c D_c^{-1}$ is $\left[(K_p + \frac{K_d s}{\tau_d s+1})\frac{s}{s+a} + \frac{K_i}{s+a}\right]\left[\frac{s}{s+a} I_r\right]^{-1}$. Since C_{pid} stabilizes G_Λ, $M_\Lambda = Y\, D_c + X_\Lambda\, N_c$ is unimodular, which implies $\text{rank} M_\Lambda(0) = r = \text{rank} X(0)\, K_i$. Therefore, $\text{rank} X(0) = r$, equivalently, G has no transmission-zeros at $s = 0$, and $\text{rank} K_i = r$. **b)** Suppose that G_Λ is stabilized by C_p, equivalently $H_p = G_\Lambda(I + C_p G_\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$; or by C_d, equivalently $H_d = G_\Lambda(I + C_d G_\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$; or by C_i, which implies $H_i = G_\Lambda(I + C_i G_\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$. The (normal) ranks of H_p, H_d, H_i are equal to $\text{rank} G = r$. By Proposition 2-(i), there exists a a P-controller for H_d, for H_i, and for H_{id}; there exists a D-controller for H_p, for H_i, and for H_{pi}. By Proposition 2-(ii), there exists an I-controller for H_p, for H_d, and for H_{pd}. Consider $H_p \in \mathcal{M}(\mathcal{H}_\infty)$: If G has no transmission-zeros at $s = 0$, then $\text{rank} H_p(0) = \text{rank}(Y + X_\Lambda C_p)^{-1}(0) X_\Lambda(0) = \text{rank} X(0) = r$. Let C_{dh} be a D-controller and C_{ih} be an I-controller for H_p. By Lemma 1, the PD-controller $C_{pd} = C_p + C_{dh}$ and the PI-controller $C_{pi} = C_p + C_{ih}$ stabilize G_Λ. Similarly, consider $H_d \in \mathcal{M}(\mathcal{H}_\infty)$: Since $M_{d\Lambda} := (Y + X_\Lambda C_d)$ is unimodular, $\text{rank} M_{d\Lambda}(0) = \text{rank} Y(0) = r$; i.e., G has no poles at $s = 0$. If G has no transmission-zeros at $s = 0$, then $\text{rank} H_d(0) = \text{rank} M_{d\Lambda}^{-1}(0) X_\Lambda(0) = \text{rank} X(0) = r$. Let C_{ph} be a P-controller and C_{ih} be an I-controller for H_d. By Lemma 1, the PD-controller $C_{dp} = C_d + C_{ph}$ and the ID-controller $C_{di} = C_d + C_{ih}$ stabilize G_Λ. Consider $H_i \in \mathcal{M}(\mathcal{H}_\infty)$: Let C_{ph} be a P-controller and C_{dh} be a D-controller for H_i. By Lemma 1, the PI-controller $C_{ip} = C_i + C_{ph}$ and the ID-controller $C_{id} = C_i + C_{dh}$ stabilize G_Λ. **c)** Suppose that G_Λ is stabilized by C_{pd}, equivalently $H_{pd} = G_\Lambda(I + C_{pd} G_\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$; or by C_{pi}, which implies $H_{pi} = G_\Lambda(I + C_{pi} G_\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$; or by C_{id}, which implies $H_{id} = G_\Lambda(I + C_{id} G_\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$. The (normal) ranks of H_{pd}, H_{pi}, H_{id} are equal to $\text{rank} G = r$. Consider $H_{pd} \in \mathcal{M}(\mathcal{H}_\infty)$: If G has no transmission-zeros at $s = 0$, then $\text{rank} H_{pd}(0) = \text{rank}(Y + X_\Lambda\, C_{pd})^{-1}(0) X_\Lambda(0) = \text{rank} X(0) = r$. Let C_{ih} be an I-controller for H_{pd}. Let C_{dh} be a D-controller for H_{pi}. Let C_{ph} be a P-controller for H_{id}. By Lemma 1, each of the PID-controllers $C_{pdi} = C_{pd} + C_{ih}$, $C_{pid} = C_{pi} + C_{dh}$, and $C_{idp} = C_{id} + C_{ph}$ stabilize G_Λ. □

Proof of Lemma 2: Let $G = Y^{-1}X$ be an LCF of G. Let C_{pid} be a PID-controller that stabilizes G_Λ. An RCF $C_{pid} = N_c D_c^{-1}$ is given in Proposition 3. Then $\det\, D_c(z_i) = \det\, \frac{z_i}{z_i+a} I_r > 0$ for all $z_i > 0$. If C_{pid} stabilizes G_Λ, then $M_\Lambda = Y\, D_c + X_\Lambda\, N_c$ is unimodular, which implies $\det\, M_\Lambda(z_i) = \det Y(z_i)\, \det\, D_c(z_i)$ has the same sign for all $z_i \in \mathcal{U}$ such that $X(z_i) = 0$; equivalently, $\det Y(z_i)$ has the same sign at all blocking-zeros of G. Therefore, G has the parity-interlacing-property; hence, it is strongly stabilizable, (Vidyasagar, 1985). □

Proof of Proposition 4: *i)* The controller C_{pd} stabilizes G_Λ if and only if $M_{pd} := Y + X_\Lambda\, C_{pd} = \frac{(s-p)}{as+1}[I + G_\Lambda\, C_{pd}]$ is unimodular. Now M_{pd} is unimodular if and only if $\det \frac{(s-p)}{as+1}[I + G_\Lambda\, C_{pd}] = \det \frac{(s-p)}{as+1} \det[I + C_{pd} G_\Lambda]$ is a unit in $\mathcal{H}_\infty$, and equivalently, $\widetilde{M}_{pd} := \frac{(s-p)}{as+1}[I + C_{pd} G_\Lambda] = Y + C_{pd} X_\Lambda$ is unimodular. Writing $M_{pd} = \frac{(s-p)}{as+1}[I + (\alpha + p)\, G_\Lambda\, \widehat{C}_{pd}] = [I + \frac{(\alpha+p)\, s}{s+\alpha}\, \Phi_\Lambda]\, \frac{(s+\alpha)}{(as+1)}$, a sufficient condition for M_{pd} to be unimodular is that $(\alpha + p) < \min_{\mathcal{T}\in\Theta} \parallel \Phi_\Lambda \parallel^{-1}$. Similarly, writing $\widetilde{M}_{pd} = [I + \frac{(\alpha+p)s}{s+\alpha} \widetilde{\Phi}_\Lambda]\, \frac{(s+\alpha)}{(as+1)}$, a sufficient condition for $\widetilde{M}_{pd}$ to be unimodular is that $(\alpha + p) < \min_{\mathcal{T}\in\Theta} \parallel \widetilde{\Phi}_\Lambda \parallel^{-1}$. Since M_{pd} is unimodular if and only if $\widetilde{M}_{pd}$ is unimodular, the less conservative one of these bounds suffices and hence, C_{pd} in (8) stabilizes G_Λ for α satisfying (7). *ii)* Since C_{pd} stabilizes G_Λ, $H_{pd} := M_{pd}^{-1} X_\Lambda = G_\Lambda(I + C_{pd} G_\Lambda)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$, where $H_{pd}(0)^{-1} = G^{-1}(0) + K_p = X(0)^{-1} Y(0) + (\alpha + p) X(0)^{-1} = \alpha\, X(0)^{-1}$. Using similar steps as in the proof of Proposition 2, the I-controller $K_i/s = \gamma H_{pd}(0)^{-1}/s$ stabilizes H_{pd} for any $\gamma \in \mathbb{R}$ satisfying (9). So, C_{pid} in (10) stabilizes G_Λ. □

Proof of Proposition 5: Omitted due to space restrictions. See the full version of the paper submitted for publication in Automatica.

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